On the Masses of the Physical Mesons: Solving the Effective QCD–Hamiltonian by DLCQ

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Abstract

The effective QCD-interaction as obtained from the front form Hamiltonian by DLCQ is Fourier transformed on the Retarded Schrödinger Equation for to describe the constituents of physical mesons. The crucial point is the use of a running coupling constant $\alpha_s(Q^2)$, in a manner similar but not equal to the one of Richardson in the equal usual-time quantization. Fixing the running coupling constant at the Z mass, the only parameters are the flavor masses. Without the top quark one needs thus 5 parameters to calculate the physical masses of 30 pseudoscalar and vector mesons, consistently within the same model. Applying variational methods to a caricature of the model, the biggest technical challenge is the solution of a cubic algebraic equation. – In view of an oversimplified model and a very simple technology, the agreement with the empirical data is much too good.

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1 Introduction and Motivation

One of the most outstanding tasks in strong interaction physics is to calculate the spectrum and the wavefunctions of physical hadrons from Quantum Chromodynamics. The method of 'Discretized Light-Cone Quantization' (DLCQ) [1] has precisely this goal and has three major aspects:

(1) a rejuvenation of the Hamiltonian approach,

(2) a denumerable Hilbert space of plane waves,

(3) Dirac's front form of Hamiltonian Dynamics.

In the *front form*, using Dirac's terminology [2], one quantizes at equal 'light-cone time' $x^+ = x^0 + x^3$, as opposed to the *instant form* where one quantizes at equal 'usual time' $t = x^0$. The front form carries many names like for example *light-cone quantization* [3], see also [4]. The approach has some unique features as reviewed in [4]: The vacuum is simple, or at least simplier than in the instant form, which in turn implies that the groundstate of the free theory is also the groundstate of the interacting theory. The relativistic wavefunctions transform trivially under certain boost operations [2, 4]. Both are in stark contrast to the conventional instant form. Discretization, that is the use of periodic or antiperiodic boundary conditions, provides additional length scales as useful *regularization parameters* in a formalism like a quantum field theory which otherwise is littered with divergencies *ab ovo*. In fact discretization (or working 'on a circle') is more than only a useful tool for calculations, rather it provides a systematic scheme to formulate the theory *at all*.

Over the years, the light-cone community [5] has made much progress. Calculations have been done [4] to verify agreement with other methods particularly lattice gauge theory. We have learned that *zero modes* should not be ignored [6], since they can be important carriers of quantum structure. Other calculations in 1-space and 1-time dimension like the dimensionally reduced models [7, 8] have an interest of their own. We have learned that there are physical and non-physical gauges, even something like a wavefunction of the vacuum [8]. Renormalization is a difficult problem within a Hamiltonian theory and worth the effort of the best [9, 10].

What was *not* shown thus far, however, is why all that is useful. Particularly one lacks the contact to phenomenology beyond the perturbative regime [4]. We believe that one needs more work like [11] or like [12, 13] which relate formalism and actual experimental numbers. It should be possible to combine rigor with simplicity! The present work will hopefully be a useful contribution to the extent that we shall derive a rather simple, conventional second order differential equation for the wavefunction in configuration space. Amazingly enough, it seems possible to describe quarks

in hadrons by a non-relativistic Schrödinger equation to account for the correct retardation and still being in the front form of Hamiltonian Dynamics applied to Quantum Chromodynamics.

Solving this 'Retarded Schrödinger Equation', or better a caricature of this equation, by parametric variation, we end up with the masses of all pseudoscalar and vector mesons. In comparison with the experimental data [14], they are no worse than conventional phenomenological models [15, 16], or predictions from heavy quark symmetry [17], or even predictions based on lattice gauge calculations [18, 19, 20]. In view of the simplicity of the methods applied we regard this as a success of the front-form approach, particularly since we can and will do better in the future [31].

2 The Model: The Effective QCD-Hamiltonian from DLCQ

In Discretized Light-Cone Quantization one addresses oneself to solve an eigenvalue problem

$$H_{\rm LC}|\psi_i\rangle = E_i|\psi_i\rangle \ . \tag{1}$$

The operator $H_{\rm LC} \equiv P^\mu P_\mu$ is a Lorentz-scalar with the dimension of a $<mass>^2$. The eigenvalues have the same dimensions. We therefore set $E_i = M_i^2$ and interprete M_i as the 'mass of the state i'. In DLCQ one works in momentum representation, and thus three of the four components of P^μ are diagonal, namely the three total (light-cone) momentum operators P^+ and $\vec{P}_\perp$. The fourth component, the total (light-cone) energy operator P^- is very complicated and off-diagonal. The four components of energy-momentum P^μ mutually commute; their matrix elements can be found explicitly in [4]. The eigenvalues of P^+ and $\vec{P}_\perp$ are

$$P^+ = \sum_j k_j^+ \ , \quad \text{and} \quad \vec{P}_\perp = \sum_j \vec{k}_{\perp_j} \ , \tag{2}$$

where j runs over all partons (particles) in a Fock state; each parton has a four-momentum denoted by $k_j^\mu = (k_j^-, k_j^+, \vec{k}_{\perp_j})$ and sits on its mass shell $(k^\mu k_\mu)_j = m_j^2$. Based on the boost-properties of light-cone operators [4] one always can transform to an intrinsic frame, where $\vec{P}_\perp = 0$ and thus $P^\mu P_\mu = P^+P^-$. Since P^+ is diagonal from the outset, the diagonalization of P^- and of $H_{\rm LC}$ amounts to the same problem.

The Hilbert space for diagonalizing P^- is spanned by all possible Fock states which have fixed and given eigenvalues of P^+ and $\vec{P}_\perp = (0,0)$. They can be arranged into 'sectors' $n = 1, 2, \ldots, N$, corresponding to their parton composition. We convene to call the sector with one quark and one antiquark as the sector with $n = 1$. The sector with one quark (q), one antiquark ($\bar{q}$), and one

gluon (g) shall be denoted by $n = 2$, and so on. It is peculiar to DLCQ that the number of sectors is limited for any fixed value of P^+, or of the 'harmonic resolution' $K = \pi P^+/L$ [1], since the longitudinal momenta k_j^+ are all positive and non-vanishing numbers. Both K and N can thus be arbitrarily large but are strictly finite. The eigenvalue equation, Eq.(1), can thus be written as a matrix equation for a *finite* number of block-matrices $\langle n|H|n'\rangle$, *i.e.*

$$\begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle & \dots & \langle 1|H|N\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle & \dots & \langle 2|H|N\rangle \\ \vdots & & \ddots & \vdots \\ \langle N|H|1\rangle & \langle N|H|2\rangle & \dots & \langle N|H|N\rangle \end{pmatrix} \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \vdots \\ \langle N|\psi\rangle \end{pmatrix} = M^2 \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \vdots \\ \langle N|\psi\rangle \end{pmatrix} . \tag{3}$$

The projected wavefunctions are denoted by $\langle n|\psi\rangle$. The probability amplitude for finding only a $q\bar{q}$-pair with no gluons, in a pion for example, is denoted by $\langle 1|\psi\rangle = \psi_{q\bar{q}/\pi}$, and correspondingly for the higher sectors. The Hamiltonian matrix Eq.(3) has a finite number of sectors, but thus far each sector has an infinite dimension, since the transversal momenta of each parton can take any positive or negative value, including zero. The number of states is however finite if one admits only those transversal momenta whose absolute values are smaller than a 'cut-off'. The regularization of Brodsky and Lepage, *i.e.*

$$\sum_{j\in n} \Big(\frac{m^2 + \vec{k}_\perp^{\,2}}{x}\Big)_j \leq \Lambda_n^2 \ , \qquad \text{with} \quad x_j = \frac{k_j^+}{P^+} \ , \tag{4}$$

does that in a Lorentz-invariant way. The cut-off mass Λ_n can but need not depend on the sector number. In the $q\bar{q}$ sector, we shall use below $\Lambda_1^2 = \Lambda^2 + (m_1 + m_2)^2$ such that the 'dynamic' cut-off Λ is a measure for the off-shell momenta. The Hamiltonian matrix is then strictly finite: A finite number of sectors, and within each sector a finite number of Fock states. Now, any $N \times N$ block matrix equation with the Hamiltonian H can be transformed identically [21] into a 1×1 block matrix equation with the effective Hamiltonian $H_{\rm eff}$, *i.e.*

$$H_{\rm eff}|\phi\rangle = M^2|\phi\rangle \ , \qquad \text{with} \quad |\phi\rangle = |1\rangle\langle 1|\psi\rangle \ . \tag{5}$$

As a consequence of working with the front form, the light-cone Hamiltonian [4] and therefore also the effective Hamiltonian is additive in the 'kinetic energy' T and the 'effective interaction' $\widetilde{U}$, *i.e.*

$$M^2 \ |\phi\rangle = T \ |\phi\rangle + \widetilde{U} \ |\phi\rangle \ . \tag{6}$$

Note that the effective Hamiltonian acts only in the sector with $n = 1$, *i.e.* in the $q\bar{q}$ sector. The reduction from Eq.(4) to Eqs.(5) or (6) is straightforward but complicated, involving only well-defined matrix inversions and multiplications [21]. A number of points should be emphasized: No unphysical Fock-space truncations are needed as they are advocated sometimes as Tamm-Dancoff truncations [22, 23]. No smallness assumptions on the coupling constant is ever to be made. The algorithm expresses the 'higher Fock space projections' $\langle n|\psi\rangle$ for $n \geq 2$ as a functional of $\langle 1|\psi\rangle$. Finally, one should emphasize that the procedure [21] is *only well defined* if the Fock space is denumerable like in DLCQ. As a result one ends up essentially with two contributions to the effective interaction, as displayed in Fig. 1. A more thorough presentation is presently being prepared.

Every finite and well-defined Hamiltonian matrix has thus a unique and well-defined effective Hamiltonian matrix. In the former [25] and the present work one considers only the effective interaction as displayed on the left in Fig. 1 and disregards (not neglects!) the annihilation term. Without further arguing about that we consider the so defined effective interaction as part of our model.

DLCQ was particularly useful to convert the matrix equation (3) to another matrix equation (6). Once this is done, one can leave the *discretized* case by going over to the *continuum limit*. In the continuum limit the matrix equation (6) is converted into the integral equation [25]

$$M^2\ \phi_{s_1,s_2}(x,\vec{k}_\perp) = \Big(\frac{m_1^2+\vec{k}_\perp^{\,2}}{x} + \frac{m_2^2+\vec{k}_\perp^{\,2}}{1-x}\Big)\ \phi_{s_1,s_2}(x,\vec{k}_\perp)$$
$$+ \sum_{s_1',s_2'} \int dx'\, d^2\vec{k}_\perp'\ \frac{2m_1 m_2\ \Theta'}{\sqrt{x(1-x)\,x'(1-x')}}\ \widetilde{U}^{s_1',s_2'}_{s_1,s_2}(x,\vec{k}_\perp;x',\vec{k}_\perp')\ \phi_{s_1',s_2'}(x',\vec{k}_\perp')\ . \tag{7}$$

The kernel is written conveniently in terms of the interaction and the fermion currents,

$$\widetilde{U} = \widetilde{V}\widetilde{C}\ , \qquad \text{with} \qquad \widetilde{V}(x,\vec{k}_\perp;x',\vec{k}_\perp') \;=\; -\,\frac{1}{2\pi^2}\,\frac{\beta(Q_a^2)}{Q_a^2}\ ,$$
$$\text{and} \quad \widetilde{C}^{s_1',s_2'}_{s_1,s_2}(x,\vec{k}_\perp;x',\vec{k}_\perp') \;=\; \bar{u}(k_1,s_1)\,\gamma^\mu\,u(k_1',s_1')\ \bar{u}(k_2,s_2)\,\gamma_\mu\,u(k_2',s_2')\ . \tag{8}$$

The cut-off function $\Theta' \equiv \Theta(x',\vec{k}_\perp')$ should be a remainder, that the domain of integration is limited according to Eq.(4). The four-momentum transfer Q_a^2 is defined as the average of the quark and the antiquark, *i.e.*

$$Q_a^2 = -q_a^2 = -\frac{1}{2}(k_1'-k_1)^2 - \frac{1}{2}(k_2'-k_2)^2\ . \tag{9}$$

Finally, β is the dimensionless coupling constant. For QED it takes the value $\beta = \alpha$ with $\alpha \sim 1/137$. For QCD with 3 colors it takes the value $\beta = \frac{4}{3}\alpha_s$. In either case, it depends on the momentum transfer due to the renormalization group.

The kernel of Eq.(8) looks different from [25] without being so. Here, we shall work in the Bjørken-Drell convention for the spinors such that in lowest order of approximation holds $j^\mu j_\mu \sim 1$, as opposed to the Lepage-Brodsky convention [4] were one has $j^\mu j_\mu \sim 4m_1 m_2$. In addition, a factor of $4\sqrt{x(1-x)x'(1-x')}$ is missing in the denominator due to a typing-error in [25]. – As illustrated in Fig.1, the effective interaction scatters a single quark with four-momentum k_1 and spin-projection s_1 into a quark with (k_1', s_1'), and correspondingly an antiquark from (k_2, s_2) to (k_2', s_2'). The eigenfunctions generated by Eq.(7) are identical with the probability amplitudes of the Fock-state expansion for 3 colors, *i.e.*

$$\begin{aligned} |\psi\rangle &= \sum_{s_1,s_2} \int dx\, d^2\vec{k}_\perp\, \phi_{s_1,s_2}(x,\vec{k}_\perp)\, |x,\vec{k}_\perp; s_1, s_2\rangle \\ \text{with} \quad |x,\vec{k}_\perp; s_1, s_2\rangle &= \frac{1}{\sqrt{3}} \sum_{c=1}^{3} b^\dagger_{c,s_1}(x,\vec{k}_\perp)\, d^\dagger_{c,s_2}(1-x,-\vec{k}_\perp)\, |0\rangle \ . \end{aligned} \tag{10}$$

They depend on five entries: the three momenta of the quark and the two spin projections. One may or may not consider Eq.(7) as the point of orign for the present model. The principal aim of the present work is to generate solutions or reasonable approximations thereto.

Having been so explicit, the spin projections and their summation will be suppressed in the sequel as being performed implicitly. The integral equation can then be written more compactly as

$$M^2\phi(x,\vec{k}_\perp) = \Big(\frac{m_1^2+\vec{k}_\perp^{\,2}}{x} + \frac{m_2^2+\vec{k}_\perp^{\,2}}{1-x}\Big)\phi(x,\vec{k}_\perp) + \int dx'\, d^2\vec{k}_\perp' \frac{2m_1 m_2 \Theta'\, \widetilde{U}(x,\vec{k}_\perp; x',\vec{k}_\perp')}{\sqrt{x(1-x)\,x'(1-x')}} \phi(x',\vec{k}_\perp') \ . \tag{11}$$

In practice it is inconvenient to deal with the invariant mass-squared eigenvalues since binding effects are so much overshadowed by the large value of the total free mass $\overline{M} = m_1 + m_2$. It is therefore suggestive to substitute the light-cone Hamiltonian $H_{\rm LC}$ by an other operator H, which differs by the former only by a constant and a readjustment of scale, *i.e.*

$$H_{\rm LC} = (m_1+m_2)^2 + 2(m_1+m_2)\ H \ . \tag{12}$$

The 'Hamiltonian' H will then have the dimension of a $\langle mass\rangle$ or $\langle energy\rangle$; in fact, as we shall see, it will have much in common with a non-relativistic Hamiltonian without being one. Since $(m_1+m_2)^2$ is a Lorentz scalar, the approach continues to be frame independent. All what one does

is to remodel the kinetic energy by

$$T \equiv T(x,\vec{k}_\perp) = \frac{1}{2(m_1+m_2)}\left(\frac{m_1^2+\vec{k}_\perp^{\,2}}{x}+\frac{m_2^2+\vec{k}_\perp^{\,2}}{1-x}-(m_1+m_2)^2\right)\ . \tag{13}$$

Without changing the substance, the integral equation becomes then

$$E\ \phi(x,\vec{k}_\perp) = T(x,\vec{k}_\perp)\ \phi(x,\vec{k}_\perp) + \int d^2\vec{k}'_\perp \frac{m_r\ dx'}{\sqrt{x(1-x)\,x'(1-x')}}\ \Theta'\,\widetilde{U}(x,\vec{k}_\perp;x',\vec{k}'_\perp)\ \phi(x',\vec{k}'_\perp)\ . \tag{14}$$

The reduced mass m_r is given by $m_r = m_1 m_2/(m_1+m_2)$, as usually. The eigenvalue of invariant mass-squared M^2 and the eigenvalue of the energy E are then related by $M^2 = \overline{M}^{\,2} + 2\overline{M}E \neq (\overline{M}+E)^2$, distinctly different from non-relativistic kinematics.

The running coupling constant does not depend on whether one calculates it perturbatively in the usual coordinates or by light-cone methods and behaves for sufficiently large momentum transfer like [27, 28]

$$\alpha_s(Q^2) = \frac{12\pi}{33-2n_f}\,\frac{1}{\ln(Q^2/\kappa^2)}\ , \qquad \text{for} \quad Q^2 \gg \kappa^2\ . \tag{15}$$

With the value of α_s as measured at the Z-mass [14] and three flavours ($n_f = 3$), one gets

$$\alpha_s(M_Z) = 0.1134 \pm 0.0035\ , \qquad \text{thus} \quad \kappa = 193\,\text{MeV}\ . \tag{16}$$

The value of the 'QCD-scale' κ is thus fixed and will not be changed during this work. But here is the problem: In a Hamiltonian approach like the present one, one needs the interaction not only at large but also at small momentum transfer, including zero. Taking the above functional at face value implies that hadronic interactions are *repulsive* for sufficiently small momentum transfer. This must plainly be wrong, of course, and the running must be modified. – But how? – Neither canonical renormalization theory nor the present DLCQ-formalism has a ready-to-use answer, yet. We therefore modify it in the simplest possible way, inserting a constant at the right place:

$$\alpha_s(Q^2) = \frac{12\pi}{27}\,\frac{1}{\ln(a^2+Q^2/\kappa^2)} \qquad\qquad a \simeq 1\ . \tag{17}$$

Right or wrong, it does what one wants: The asymptotic behaviour is untouched and α_s keeps its sign all the way down to $Q^2 = 0$. A similar step was taken by Richardson [24] for the instant form, but the value of κ was considered a free parameter. In the sequel we shall follow Richardson setting $a = 1$, we shall however go much beyond that by including the correct fine- and hyperfine interaction.

The change from Eq.(15) to Eq.(17) is the only assumption in the present work: Despite not completely true, one could say that 'confinement is introduced by hand' like in conventional potential models. The crucial difference to other models is that the correct asymptotic freedom [27, 28] is combined with the virtue of having no free parameter and some further pleasant aspects:

- The model depends only on the *quark currents* and is therefore manifestly gauge-invariant.
- The model is formulated in the front form and therefore frame- and boost-invariant.
- The usual 'recoil problem' is absent in a momentum representation.

Finally, one should mention here already that the Light-Cone formalism has a different operator structure than the conventional instant form: The total angular momentum J and its eigenvalues cannot be used to classify the eigenstates [4]. The non-initiated reader might find it quite surprizing that the wavefunctions *can never be invariant* under spatial rotations about the y- or the x-axis.

3 Analyzing the Model by Transforming Integration Variables

Like in other models, the flavor quark masses are free parameters subject to be determined by comparing with experiment. Natural candidates are the pseudoscalar (0^-) and vector mesons (1^-), which are believed to have total angular momentum $L = 0$. Their total spin is thus $J = S$. For two quarks, the total quark spin S can only take the value $S = 0$ or $S = 1$. Since the flavor quark masses potentially range from a few MeV up to some 100 GeV, one faces a tantalizing numerical problem: For any given set of quark and antiquark masses, m_1 and m_2, respectively, one has to solve Eq.(7) and to perform a calculation of similar complexity as in [25]. Actually, in addition to that, one is confronted with thus far unresolved, technically highly non-trivial problems due to a Q^4 singularity in the denominator. Moreover, of all the calculated *spectrum* one only needs the *lowest* eigenvalue. Only the latter will be compared with the physical meson mass M. Most of the calculated and thus expensive information is thus redundant, not to mention that the calculation will have to be repeated a zillion of times in order to fit m_1 and m_2 to M. — Is there a simplier way? The following analysis will pave the way for an ultimate, possibly analytical solution.

The fundamental equation is unpleasant to the practitioner. In Eq.(7) two of the three integration variables are dimensionful ($\vec{k}_\perp$) and run from $-\infty$ to $+\infty$, while the third one is dimensionless and has support only on the finite domain $0 \leq x \leq 1$. It is therefore more than convenient to

change variables from x to k_z, such that k_z has the same support and dimension as $\vec{k}_\perp$. This can be achieved by

$$x = x(k_z) = \frac{k_z + E_1}{E_1 + E_2} \ , \qquad \text{with} \quad E_i = \sqrt{m_i^2 + \vec{k}_\perp^{\,2} + \vec{k}_z^{\,2}} = \sqrt{m_i^2 + \vec{k}^{\,2}} \quad \text{for} \quad i = 1, 2 \ . \tag{18}$$

The physical interpretation of k_z is of secondary importance. One notes with pleasure that $k_z = 0$ and $\vec{k}_\perp = 0$ produces the correct 'static value' $\bar{x} = m_1/(m_1 + m_2)$ and that one can formally define a three-vector $\vec{k} = (k_z, \vec{k}_\perp)$. The practitioner therefore will conclude that k_z is 'something like' the relative momentum in z-direction. The advantages are obvious, particularly the notation is more compact than in the standard front form. For example, the free part of the invariant mass squared appearing in Eqs.(4) and (7) becomes by direct substitution simply

$$\frac{m_1^2 + \vec{k}_\perp^{\,2}}{x} + \frac{m_2^2 + \vec{k}_\perp^{\,2}}{1 - x} = (E_1 + E_2)^2 \ . \tag{19}$$

It looks as if one changes the single particle four-momentum state from the front form with $k_1^\mu = (k_1^-, k_1^+, \vec{k}_{\perp_1})$ and $k_2^\mu = (k_2^-, k_2^+, \vec{k}_{\perp_2})$, to a state in the instant form with

$$k_1^\mu = (k_1^0, \vec{k}_{\perp_1}, k_1^3) = (E_1, \vec{k}) \ , \qquad \text{and} \quad k_2^\mu = (k_2^0, \vec{k}_{\perp_2}, k_2^3,) = (E_2, -\vec{k}) \ . \tag{20}$$

The transfer and the mean of three-momentum along the quark line is then given by

$$\vec{q} = \vec{k} - \vec{k}' \qquad \text{and} \quad \vec{p} = \frac{1}{2}(\vec{k} + \vec{k}') \ , \tag{21}$$

respectively. Along the antiquark line, both have the opposite sign. But all this does *not imply* that one works in the instant form. Indeed, one has explicit residues from the front form, as follows. Because of boost-invariance, Eq.(7) is independent of the total longitudinal momentum P^+. But P^+ still lurks in the background, since the Fock-states 'on the left' ($|x, \vec{k}_\perp\rangle$) have the same P^+ than those 'on the right' ($|x', \vec{k}'_\perp\rangle$). One can express this fact in instant form variables as well, using $P^+ = P^0 + P^3$. Since $P^3 = k_{z,1} + k_{z,2} = 0$, one is left with $P^0 = P^{0\prime}$, and therefore has for every matrix element:

$$E_1 + E_2 = E_1' + E_2' \ , \qquad \text{or} \qquad \vec{k}^2 = \vec{k}'^{\,2} \ . \tag{22}$$

Because of the latter, one has also individually $E_1 = E_1'$ and $E_2 = E_2'$. Obviously, the interaction *does not change the size* of $\vec{k}$, it only changes its direction: The frame-independent front-form formalism is equivalent to the center-of-mass frame of the instant form. This, of course, is a source

of great simplification. For example, the momentum transfer and the mean momentum are always orthogonal,

$$\vec{q}\cdot\vec{p}=\frac{1}{2}(\vec{k}-\vec{k}')(\vec{k}+\vec{k}')=\frac{1}{2}(\vec{k}^{\,2}-\vec{k}'^{\,2})=0\ , \tag{23}$$

and four-momentum transfer is always identical with the three-momentum transfer. The latter is verified easily. One inserts the single four-momenta from Eq.(19) into the defining Eq.(9) and gets

$$Q_a^2=\frac{1}{2}\left[(\vec{k}_1'-\vec{k}_1)^2-(E_1'-E_1)^2+(\vec{k}_2'-\vec{k}_2)^2-(E_2'-E_2)^2\right]=\vec{q}^{\,2}\ , \tag{24}$$

This way, the interaction $\widetilde{V}$ depends *only* on the spatial momentum-transfer.

It therefore must be our aim to express all functions appearing in the light-cone integral equation (14) as functions of the instant form variables $\vec{k}$ and $\vec{k}'$, *i.e.* to identically rewrite it as

$$E\,\phi(\vec{k}\,)=T(\vec{k}\,)\,\phi(\vec{k}\,)+\int d^3\vec{k}'\ \Theta(\vec{k}')\,\widetilde{A}(\vec{k},\vec{k}')\,\widetilde{B}(\vec{k},\vec{k}')\,\widetilde{C}(\vec{k},\vec{k}')\,\widetilde{V}(\vec{k},\vec{k}')\,\phi(\vec{k}\,')\ . \tag{25}$$

The notation can be chosen such that the Jacobian $\widetilde{B}$ and the current term $\widetilde{C}$ are dimensionless with value '1' by lowest order of approximation, as will be unrevealed gradually. Moreover, rather than by $\vec{k}$ and $\vec{k}'$, we shall express them often by the above $\vec{q}$ and $\vec{p}$, since they have a very special meaning in integral equations like Eq.(25), as follows. Rewrite the latter conveniently as $E\phi(\vec{k})=\int d^3\vec{k}\,'\,\widetilde{F}(\vec{q},\vec{p})\ \phi(\vec{k}\,')$. Multiply the whole equation with $\exp(i\vec{k}\cdot\vec{x})$ and subsequently integrate over $d^3\vec{k}$. Making use of the definition

$$\psi(\vec{x}\,)=\int d^3\vec{k}\ \mathrm{e}^{i\vec{k}\cdot\vec{x}}\ \phi(\vec{k}\,')\ ,\qquad \text{and thus}\quad F(\vec{x},\vec{p}\,)=\int d^3\vec{q}\ \mathrm{e}^{i\vec{q}\cdot\vec{x}}\ \widetilde{F}(\vec{q},\vec{p}\,)\ , \tag{26}$$

one straightforwardly arrives at an eigenvalue equation of the *Schrödinger type*, *i.e.*

$$E\psi(\vec{x})=F(\vec{x},\vec{p}\,)\ \psi(\vec{x})\ ,\qquad \text{with}\quad \vec{p}\equiv-i\vec{\nabla}_x\ . \tag{27}$$

Thus, the analogue of $\vec{q}$ is $\vec{x}$, the position of the particle in the center-of-mass frame, and $\vec{p}$ is its conjugate momentum. Consider for example the interaction term in Eq.(25). Since $\widetilde{B}=\widetilde{C}=1$ will hold to lowest order, see below, one gets

$$\widetilde{V}(\vec{q}^{\,2})=-\frac{1}{2\pi^2}\frac{\alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}}\ ,\qquad \text{thus}\quad V(\vec{x}\,)=\frac{8\pi}{27}\Big(-\frac{1}{r}+\kappa^2\,r\Big)\ . \tag{28}$$

The Fourier transform generates a Coulomb potential plus a confining potential with linearly rising walls, to a high degree of accuracy [24]. This potential is plotted in Fig. 2 versus $r=|\vec{x}|$.

Let us begin the programme with the kinetic energy $T(\vec{k}\,) = \left((E_1+E_2)^2 - \overline{M}^2\right)/(2\overline{M})$. It is trivially a function only of $\vec{k}$. The cut-off function $\Theta' = \Theta(\vec{k}\,')$ is related to the mass-scale Λ introduced in Eq.(4) to render the Hamiltonian matrix finite. The latter becomes now $(E_1+E_2)^2 \leq \Lambda^2 + (m_1+m_2)^2$, which in turn can be cast identically into

$$\vec{k}^{\,2} \leq \left(\frac{\Lambda}{2}\right)^2 \frac{\Lambda^2 + 4m_1m_2}{\Lambda^2 + (m_1+m_2)^2}\ . \tag{29}$$

As expected, the cut-off is invariant under spatial rotations. Next, using the Gordon decomposition, the currents $\bar{u}(k_i,s_i)\,\gamma^\mu\,u(k_i',s_i')$ are conveniently decomposed into their convective and spin parts,

$$\bar{u}(k_1,s_1)\,\gamma^0\,u(k_1',s_1') \;=\; \frac{E_1+m_1}{2m_1}\left(1 + \frac{\vec{p}^{\,2} - \vec{q}^{\,2}/4}{(E_1+m_1)^2} + \frac{\vec{R}\cdot\vec{\sigma}}{(E_1+m_1)^2}\right)_{s_1,s_1'}\ , \tag{30}$$

$$\bar{u}(k_1,s_1)\,\vec{\gamma}\,u(k_1',s_1') \;=\; \frac{1}{2m_1}\,(2\vec{p} - i\,\vec{q}\times\vec{\sigma})_{s_1,s_1'}\ , \qquad \text{with} \quad \vec{R} = i\vec{q}\times\vec{p}\,. \tag{31}$$

Due to the validity of Eq.(22) these expressions are simpler than usually. Along the antiquark line one must change the sign of both $\vec{k}$ and $\vec{k}'$ and can conveniently substitute σ by τ in order not to confuse the spin of quark and antiquark. The current term $\widetilde{C} = j^\mu j_\mu$ becomes then explicitly

$$\begin{aligned}\widetilde{C}(\vec{k},\vec{k}'\,) \;&=\; \widetilde{D}\left(1 + \frac{\vec{p}^{\,2} - \vec{q}^{\,2}/4}{(E_1+m_1)^2} + \frac{\vec{R}\cdot\vec{\sigma}}{(E_2+m_2)^2}\right)\left(1 + \frac{\vec{p}^{\,2} - \vec{q}^{\,2}/4}{(E_1+m_1)^2} + \frac{\vec{R}\cdot\vec{\tau}}{(E_2+m_2)^2}\right) \\ &+\; \widetilde{D}\left(\frac{2\vec{p} - i\,\vec{q}\times\vec{\sigma}}{E_1+m_1}\right)\left(\frac{2\vec{p} - i\,\vec{q}\times\vec{\tau}}{E_2+m_2}\right)\ , \quad \text{with } \widetilde{D} = \frac{(E_1+m_1)(E_2+m_2)}{4m_1m_2}\ .\end{aligned} \tag{32}$$

As one expects for a Lorentz scalar it is invariant under spatial rotations. Finally, the Jacobian of the transformation Eq.(18) is evaluated in terms of the dimensionless transformation function $\widetilde{B}(\vec{k},\vec{k}'\,)$, implicitly defined by

$$\frac{m_r}{\sqrt{x(1-x)\,x'(1-x')}}\;dx' = \widetilde{A}(\vec{k},\vec{k}'\,)\widetilde{B}(\vec{k},\vec{k}'\,)\;dk_z\ . \tag{33}$$

By means of the identities

$$\frac{\partial x}{\partial k_z} = \frac{(E_1+k_z)(E_2-k_z)}{E_1E_2\,(E_1+E_2)} \qquad \text{and} \qquad x(1-x) = \frac{(E_1+k_z)(E_2-k_z)}{(E_1+E_2)^2}\ , \tag{34}$$

one gets straightforwardly

$$\widetilde{A}(\vec{k},\vec{k}\,') = \sqrt{\frac{(E_1+k_z')(E_2-k_z')}{(E_1+k_z)(E_2-k_z)}}\ , \qquad \text{and} \quad \widetilde{B}(\vec{k},\vec{k}\,') = \left(\frac{m_r}{E_1} + \frac{m_r}{E_2}\right)\ . \tag{35}$$

Here $\widetilde{A}$ is the only term which explicitly 'breaks' rotational invariance since only here appears k_z explicitly. This is no failure, but must be true for any front form Hamiltonian. As a consequence,

the wavefunction of the singlet state, for example, cannot be strictly invariant under rotations in three-space. One even has numerical evidence for such a 'violation' due to the pioneering work of Krautgärtner [25]: The singlet wavefunction displayed in his Fig. 11 show them clearly. First taken as a numerical artifact, they are meanwhile confirmed by an independent and improved calculation [26]. But these 'violations' occur at a momentum scale much larger than the Bohr momentum, in a region where the wavefunction is down typically to $10^{-3} - 10^{-4}$ of its peak value. They seem to be unimportant.

Having defined all terms in Eq.(14), it should be emphasized strongly that its solutions *must be identical* with those obtained from the original light-cone equation (7), of course after having substituted k_z back by x according to Eq.(18). It is however much simpler to deal with from the point of view of interpretation, of approximation, and last not least from the calculational point of view particularly for $m_1 \neq m_2$.

4 The Retarded Schrödinger Equation

The instant form has the garstly property of having square-roots scattered all over the place. This is particularly unpleasant when one aims at Fourier transforms. Can one find a systematic approximation scheme to remove them? – Once more this is possible by means of the Brodsky-Lapage cut-off. Indeed, one always can choose Λ such that

$$\frac{\vec{k}^{\,2}}{m_1^2} \leq 1 \ , \qquad \text{for} \quad m_1 \leq m_2 \ . \tag{36}$$

If this holds for the lighter particle it even more so holds for the heavier one. All square-roots can then be expanded safely, *i.e.*

$$E_i \simeq m_i + \frac{\vec{k}^{\,2}}{2m_i} \ , \qquad \text{if} \quad \frac{1}{8}\left(\frac{\vec{k}^{\,2}}{m_i^2}\right)^2 \ll 1 \ . \tag{37}$$

This looks like a non-relativistic approximation but the opposite is true: In the worst case, our choice allows for the ultra-relativistic velocities of the lighter particle up to $|\vec{k}\,| \sim m_1$ on the one hand, and on the other hand for a consistent and systematic expansion up to second order like

$$\widetilde{A} \ = \ 1 + \frac{aq_z}{2m_r} + \frac{a^2q_z^2}{8m_r^2} + \frac{q_zp_z}{2m_r^2}\left(a^2 + \frac{2m_r}{\overline{M}}\right) \ , \qquad \text{with} \quad a = \frac{m_1 - m_2}{m_1 + m_2} \ , \tag{38}$$

$$\widetilde{B} \ = \ 1 - \frac{\vec{p}^{\,2}}{2m_q^2} - \frac{\vec{q}^{\,2}}{8m_q^2} \ , \qquad \text{with} \quad \frac{1}{m_q^2} = \frac{1}{\overline{\overline{M}}}\left(\frac{m_2}{m_1^2} + \frac{m_1}{m_2^2}\right) \ , \tag{39}$$

$$\widetilde{D} \;=\; 1+\frac{\vec{p}^{\,2}}{4m_a^2}+\frac{\vec{q}^{\,2}}{16m_a^2}\;, \qquad\qquad \text{with}\quad \frac{1}{m_a^2}=\frac{1}{m_1^2}+\frac{1}{m_2^2}\;, \tag{40}$$

respectively. The expansion of the current term $\widetilde{C}$ of Eq.(32) and of the product of the current with the Jacobian run analogously and give

$$\widetilde{C} \;=\; 1+\frac{\vec{p}^{\,2}}{2}\left(\frac{1}{m_a^2}+\frac{2}{m_1m_2}\right)-\frac{(\vec{\sigma}\times\vec{q})\cdot(\vec{\tau}\times\vec{q})}{4m_1m_2}+\frac{\vec{\sigma}\cdot\vec{R}}{4m_1^2}+\frac{\vec{\tau}\cdot\vec{R}}{4m_2^2}-\frac{\vec{S}\cdot\vec{R}}{m_1m_2}\;, \tag{41}$$

$$\widetilde{B}\widetilde{C} \;=\; 1+\frac{\vec{p}^{\,2}}{2m_1m_2}-\frac{\vec{q}^{\,2}}{8m_q^2}-\frac{(\vec{\sigma}\times\vec{q})\cdot(\vec{\tau}\times\vec{q})}{4m_1m_2}+\frac{\vec{\sigma}\cdot\vec{R}}{4m_1^2}+\frac{\vec{\tau}\cdot\vec{R}}{4m_2^2}-\frac{\vec{S}\cdot\vec{R}}{m_1m_2}\;, \tag{42}$$

respectively. For completeness we finally define $\vec{S}$, the kinetic energy T, and Θ':

$$\vec{S}=\frac{1}{2}(\vec{\sigma}+\vec{\tau})\;, \qquad T=\frac{\vec{k}^{\,2}}{2m_r}\;, \qquad \text{and}\quad \Theta(\vec{k}\,')=1\;. \tag{43}$$

The last step needs a comment. After having expanded in the manner of Eq.(37) one has to verify in principle, that the results depend on the cut-off Λ at most weekly. One can conjecture however that the wavefunction decays sufficiently fast as to act itself like a cut-off. One thus can drop Λ at this stage. All one has to do at the end of a calculation is to check the condition set in Eq.(37), but this can and will be done by an expectation value.

One now is prepared to Fourier transform the whole Eq.(25) into an eigenvalue equation of the Schrödinger type

$$H\psi(\vec{x}\,)=E\psi(\vec{x}\,)\;. \tag{44}$$

When transforming a momentum-space function like $\vec{R}\widetilde{V}=i(\vec{q}\times\vec{p}\,)\widetilde{V}(\vec{q})$ one gets in a first step $(\vec{\nabla}\times\vec{p}\,)V$. By definition, the operator $\vec{\nabla}$ can act *only* on the next V to the right. Since V is spherically symmetric one sets $\vec{\nabla}V=\dfrac{\vec{x}}{r}\dfrac{\partial V}{\partial r}$ and absorbs $\vec{x}$ into the angular momentum operator $\vec{L}=\vec{x}\times\vec{p}$. The Hamiltonian operator H in 'Schrödinger representation' (with $\vec{p}=-i\hbar\vec{\nabla}$) turns then out as

$$\begin{aligned} H \;&=\; \frac{1}{2m_r}\left(1+\frac{V(r)}{m_1+m_2}\right)\vec{p}^{\,2}+V(r)+\frac{\vec{\nabla}^2V(r)}{8m_q^2}+\frac{(\vec{\sigma}\times\vec{\nabla})\cdot(\vec{\tau}\times\vec{\nabla}V)}{4m_1m_2} \\ &+\; \frac{1}{r}\frac{\partial V}{\partial r}\left(\frac{\vec{\sigma}\cdot\vec{L}}{4m_1^2}+\frac{\vec{\tau}\cdot\vec{L}}{4m_2^2}-\frac{\vec{S}\cdot\vec{L}}{m_1m_2}\right)+H_a\;. \end{aligned} \tag{45}$$

For a spherical V holds $(\vec{\sigma}\times\vec{\nabla})\cdot(\vec{\tau}\times\vec{\nabla}V)=\frac{2}{3}(\vec{\sigma}\cdot\vec{\tau})\vec{\nabla}^2V$ and with $\vec{\sigma}\cdot\vec{\tau}=2\vec{S}^{\,2}-3$ one arrives at

$$\begin{aligned} H \;&=\; \frac{1}{2m_r}\left(1+\frac{V(r)}{m_1+m_2}\right)\vec{p}^{\,2}+V(r)-\frac{\vec{\nabla}^2V}{8m_r^2}\left(\frac{3m_r}{m_1+m_2}-\frac{(m_1-m_2)^2}{(m_1+m_2)^2}\right)+\frac{\vec{\nabla}^2V}{3m_1m_2}\vec{S}^{\,2} \\ &+\; \frac{1}{r}\frac{\partial V}{\partial r}\left(\frac{\vec{\sigma}\cdot\vec{L}}{4m_1^2}+\frac{\vec{\tau}\cdot\vec{L}}{4m_2^2}-\frac{\vec{S}\cdot\vec{L}}{m_1m_2}\right)+H_a\;. \end{aligned} \tag{46}$$

The rotationally asymmetric H_a is given below. We emphasize here already that the precise form of V need not be known in performing the step from Eq.(42) to Eq.(46). If one sticks to QCD in the above form, Eq.(28), one has

$$\vec{\nabla}^2 V(r) = \beta \left(\frac{2\kappa^2}{r} + 4\pi\delta(\vec{x}) \right) \qquad \text{and} \quad \frac{1}{r}\frac{\partial V(r)}{\partial r} = \beta \left(\frac{\kappa^2}{r} + \frac{1}{r^3} \right) , \qquad \text{with} \quad \beta = \frac{8\pi}{27} \simeq 0.93 \ . \tag{47}$$

If one sticks to QED, one chooses $\beta \simeq 1/137$ and sets the QCD-scale κ to zero. These equations are at the core of the present work. They are still identical with the original equations (14) or (25) and include the correct retardation without smallness asumptions particularly on the coupling constant or the mass parameters.

The 'Retarded Schrödinger Equation' has a wonderfully simple structure and can be interpreted with ease. One notes that the 'potential' V plays a different role in the different terms of the equation. The first term is the kinetic energy. Like in non-relatistic quantum mechanics the reduced mass governs the scale. Here the potential plays the role of an 'effective mass'. In the second term V appears in its natural role as a potential energy with an attractive Coulomb and a linear string potential. In the second last term, the analogue of the 'fine interactions' of atomic physics, the potential plays the role of a coupling constant for the spin-orbit interaction. Last not least, in the third and the fourth term one finds the analogues of the atomic 'hyperfine interaction' with the difference that they become of leading importance in the hadronic case. They contribute in the third term to the true potential energy and for example modify the 'Coulomb' part of V, while the coefficient of $\vec{S}^2$ can be interpreted as a (color-) magnetic interaction of the two spins.

One should note that some parts of the Retarded Schrödinger equation are a direct consequence of working in the front form. We have tried to account for that in the notation. In Eq.(41), for example, one notes that the current term has no contribution in $\vec{q}^{\,2}$, while Eq.(42) does contain a term like that. They are caused by the explicit appearance of the light-cone momentum fraction x, *i.e.* are caused by the Jacobian $\widetilde{B}\widetilde{A}$. The asymmetry Hamiltonian H_a has the same orign, and following Eq.(38) has the Fourier transform

$$H_a = \frac{a_z}{2im_r}\frac{\partial V}{\partial z} - \frac{a^2}{8m_r^2}\frac{\partial^2 V}{\partial z^2} + \frac{\partial V}{\partial z}\left(a^2 + \frac{2m_r}{\overline{M}} \right)\frac{p_z}{2m_r^2} \ . \tag{48}$$

It looks like a self-induced hadronic interaction of the quadrupole type.

How serious is the Retarded Schrödinger Equation to be taken? – We have actually no argument what could have been done wrong in the above, certainly nothing on purpose or with an unjustified

assumption. At the very bottom the omission of zero-modes and, perhaps more important, the omission of the two-gluon effective annihilation graph might perhaps be influential, and has to be investigated in the future. But not even the 'Richardson potential' can be crucial. Small distances $\vec{x} \to 0$ correspond to large momenta in Fourier space, and at least there asymptotic freedom must be correct. In the opposite limit, at very large distances corresponding to small momenta in Fourier space, the detailed form of the running coupling constant might possibly matter. But there, the bound-state wavefunction is small from the outset and possible effects are unlikely to become important. Not even the consistent expansion to second order in the momenta is an assumption, since it can and must be checked *a posteriori* by

$$\frac{1}{8}\left(\frac{\langle\vec{p}^{\,2}\rangle}{m_i^2}\right)^2 \ll 1 \ . \tag{49}$$

One thus has to live with these equations. Rather than to agonize further on these almost philosophical matters, to which we shall come back in the summary, we proceed now by asking for practical consequences.

5 Solving the Model by Parametric Variation

It will take some time and effort to work out all the many consequences of the Retarded Schrödinger Equation (46), which as mentioned should hold both for QED and QCD. In the sequel, we therefore shall restrict ourselves to calculate only the ground-state masses of the pseudoscalar and vector mesons. If one leaves aside the recently discovered top-quark t and restricts to 5 flavors $(u, d; s, c; b)$, one has thus 30 different physical mesons, since charge-conjugate hadrons have of course the same mass.

One cannot calculate these masses, however, without knowing the mass parameters m_1 and m_2 of the quarks. These cannot be measured in a model-independent way, although some people pretend one can do so. Here, we shall adopt the point of view that the quark masses have to be determined consistently within each model, for the better or the worse. One has thus 5 mass parameters to account for 30 physical masses. Which ones should be selected to fit? — There are 142 506 different possibilities to select 5 members from a set of 30, and we have to make a choice: We choose the five pure $q\bar{q}$-pairs. Even that is not unique: Shall one take the pseudoscalar (0^-) or the vector mesons (1^-)? – We shall take both, and compare the results.

But doing so, one runs into the problem of choosing the chiral composition of the physical hadrons. One would be rather surprized if the pions, for example, could be understood as eigenstates in a simple potential model like Retarded Schrödinger Equation (46). Thus, in the aim to avoid hell's kitchen by choosing a certain flavor composition, we shall do even worse and replace the 'pions', for example, by 'quasi-pions' with the same physical mass: All our hadrons shall be pure $q\bar{q}$-pairs – *by fiat*. For example, we shall identify the $u\bar{u}$-, $u\bar{d}$-, $d\bar{u}$-, or $d\bar{d}$-eigenstates with the quasi-π^0, quasi-π^+, quasi-π^-, or the quasi-η, and so on. These 'crimes' can be revoked easily in future though not the present work. They are by no means compulsory to the model, as we shall see.

Our problems lie in another ball park. It is clear that we cannot deal head-on with the full complexity of the Retarded Schrödinger Equation (46). Which part of the Hamiltonian should one select in a first assault? – Some help is gained by the rather unique property of the light-cone Hamiltonian: Kinetic energy and interaction are additive, and so is the Hamiltonian. Because of this additivity one always can select an 'interesting part' H_0, *i.e.* $H = H_0 + \Delta H$ and check *a posteriori*, by calculating $\langle\, \Delta H \,\rangle$ in perturbation theory, whether the selection makes sense, or not.

Since the scalar and pseudoscalar mesons have probably no orbital excitations, *i.e.* are primarily s-waves, one can disregard the spin-orbit couplings and choose

$$H_0 = \left(1 + \frac{V(r)}{m_1 + m_2}\right) \frac{\vec{p}^{\,2}}{2m_r} + V(r) - \frac{\vec{\nabla}^2 V}{8m_r^2} \left(\frac{3m_r}{m_1 + m_2} - \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2}\right) + \frac{\vec{\nabla}^2 V}{3m_1 m_2} \vec{S}^2 \ . \tag{50}$$

Even that looks to complicated for a start-up. We therefore select those terms which in one way or the other have turned out to be important in past phenomenological work, namely a simple central potential plus the triplet-interaction mediated by the total spin. Our first choice is therefore

$$H_0 = \frac{\vec{p}^{\,2}}{2m_r} + V(r) + \frac{2\kappa}{3m_1 m_2} \frac{\vec{S}^2}{r} = \frac{\vec{p}^{\,2}}{2m_r} - \frac{\beta}{r} + \frac{2\kappa^2}{3m_1 m_2} \frac{\vec{S}^2}{r} + \beta\kappa^2 r \ . \tag{51}$$

It is always possible to choose a spinorial representation in which S_z and $\vec{S}^2$ become diagonal. In this case, the latter can be replaced by the eigenvalue $S_e = S(S+1)$, taking the values 0 and 2 for the singlet and the triplet, respectively.

How does the wavefunction for the lowest state look like? – For a pure Coulomb potential, *i.e.* for $\kappa = 0$, the solution has the form

$$\psi(\vec{x}) = \frac{1}{\sqrt{\pi}} \lambda^{\frac{3}{2}}\, \mathrm{e}^{-\lambda r} \ . \tag{52}$$

Table 1: *The flavor quark masses in MeV, as obtained from a fit to Eq.(57). — The first row refers to a fit for the singlets, the second to the one for the triplet.*

Flavor mass	u	d	s	c	b
From fit to 0^-	2.3	155.6	430.6	1642.3	5330.8
From fit to 1^-	222.8	236.2	427.2	1701.3	5328.2

One the other hand, ommitting the Coulomb part, a linear potential be solved in terms of Airy functions or its integral transforms [29]. If one has both, one will have some mixture of the two. In a numerical calculation, these can be found with almost arbitrary accuracy. But for the present start-up check even that requires too much effort.

Rather we shall pursue a variational approach. We choose Eq.(52) as a one-dimensional parameter family, subject to determine the one free parameter λ. Of course, one could take other families as well, like for example harmonic oscillator eigenstates, but with the above all expectation values are particularly simple, *i.e.*

$$\langle\psi|\,\vec{p}^{\,2}\,|\psi\rangle = \lambda^2\ , \qquad \langle\psi|\,\frac{1}{r}\,|\psi\rangle = \lambda\ , \qquad \text{and} \quad \langle\psi|\,r\,|\psi\rangle = \frac{3}{2\lambda}\ . \tag{53}$$

That's all what one needs to calculate the expectation value of the energy

$$\overline{E} = \langle\psi|H_0|\psi\rangle = \frac{\lambda^2}{2m_r} - \beta\lambda + \frac{2S_e\kappa^2}{3m_1m_2}\,\lambda + \frac{3\beta\kappa^2}{2}\frac{1}{\lambda}\ . \tag{54}$$

Since we deal only with a variational approach to the ground states, we are not in conflict with the statement that the wavefunction cannot be purely Coulombic. For the pure Coulomb case, the $2S$- and $1P$-states would be degenerate and the ratio $|\psi_{2S}(0)|^2/|\psi_{1S}(0)|^2 = 0.125$, as opposed to the values $\simeq 0.63$ and $\simeq 0.50$ extracted from experiments on charmonium and bottomium [15], respectively.

Since we address ourselves to calculate the total invariant mass of the hadrons we return to the front form invariant mass-squared operator $H_{LC} = M^2$, *i.e.* to $M^2 = (m_1+m_2)^2 + 2(m_1+m_2)\overline{E}$. Specializing now to equal masses $m_1 = m_2 = m$ one preferably converts the variational equation

Table 2: *The validity check. — The first row displays the values of* $(\lambda/m)^2$ *as obtained from the mass fit to the singlets, the second row those from the mass fit to the triplets. — Note: If* $(\lambda/m)^4 \geq 8$*, the solution has to be rejected as a consequence of Eq.(49), since* $\langle \vec{p}^{\,2} \rangle = \lambda^2$*. The extremely large value for the u-quark in pseudoscalar fit gives a good example for such a case.*

$\frac{\lambda^2}{m^2}$	u	d	s	c	b
For fit to 0^-	293	1.45	0.53	0.25	0.22
For fit to 1^-	0.79	0.76	0.48	0.25	0.22

(54) in terms of the dimensionless variables

$$s = \frac{\lambda}{\kappa} \,, \qquad \xi = \frac{m}{\kappa} \,, \qquad \text{and} \quad W = \left(\frac{M}{2\kappa}\right)^2 \,. \tag{55}$$

We measure thus all energies and masses in units of the one fixed scale in our problem, the QCD scale κ. The variational eigenvalue equation reduces then to the handy expression

$$W(s;\xi) = s^2 - \left(\beta\xi - \frac{2S_e\beta}{3\xi}\right) s + \xi^2 + \frac{3\beta\xi}{2}\,\frac{1}{s} \,. \tag{56}$$

At fixed values of the parameters (ξ, β, S_e), the variable λ and thus s must be varied such that the energy (or the mass) become stationary, *i.e.*

$$\left.\frac{\partial W(s;\xi)}{\partial s}\right|_{s=s^\star(\xi)} = 0 \,, \qquad \text{thus} \quad \left(\frac{M}{2\kappa}\right)^2 = W\left(s^\star(\xi)\right) = W^\star(\xi) \,. \tag{57}$$

This leads to a cubic equation in s which can be solved analytically in terms of Cardan's formula. In special cases they can well be approximated by a quadratic equation, namely when $s^\star \gg 1$ or when $s^\star \ll 1$. We got accustomed to refer to these *two regimes* as the *Bohr* and the *string regime*, respectively. In the Bohr regime the Coulomb potential dominates the solution and the linear string potential provides a correction. In the string regime the linear string potential dominates, with the Coulomb potential giving a correction. Solutions in the string regime, however, imply that the ratio $\langle \vec{p}^{\,2} \rangle / m^2 = (\lambda/m)^2$ becomes so large that one is in conflict with the validity condition Eq.(49).

Rather than to give explicitly the straightforward but cumbersome formalism, we present the analytical results in the graphical form of Figure 3, both for the singlet and the triplet. As displayed

there, the total mass $M = 2\kappa\sqrt{W^\star}$ is almost linear in the quark mass, with small but significant deviations. In line with expectation, the hyperfine splitting decreases with increasing quark mass. Less expected was that the splitting increases so strongly with decreasing quark mass. As one can see in the Figure, the mass of the singlet states takes off from the value 0 at vanishing mass almost like a square-root as opposed to the triplets which start off at a finite value. Therefore, if one determines the mass of the u-quark by fitting to the quasi-pion mass, as displayed in Table 1, the value of s becomes extremely small. Consequently, one is in the string regime and the u-quark becomes ultra-relativistic. The validity condition is then badly violated, as compiled in Table 2. It is worth noting that the value of the up-quark mass is rather close to what is reported for the so called 'current mass'. For the η and the η', the scaling variable s is of order unity, while for the quasi-η_c one definitely is in the Bohr regime. Here the masses are similar or close to what is referred as the 'constituent-quark' mass.

Since singlet and triplet are so close for $s \gg 1$ and since nobody wants to fit the pion anyway, quasi or not, one fits the quark masses preferentially with the vector mesons. In the lack of empirical data we have set $M_{\eta_b} = M_\Upsilon$, which should be of minor importance in the present model, see Figure 3. As seen from Table 1, the flavor masses are now in close agreement with the constituent quark masses, particularly the up- and down-quarks. As compiled in Table 2, the smallness condition is now satisfied excellently, everywhere.

Obviously, it would be a minor modification to introduce some preferred chiral structure of the light mesons, and to drop the simple and simple-minded construction of quasi-mesons. But we have an other objection, here. We ask: With the so fixed flavor quark masses, how bad or how good are the remaining 25 meson masses described? – The procedure runs quite analogously, except that it is now easier. With fixed masses, one varies λ in Eq.(54) such that the energy becomes stationary. Of course, this has to be done separately for each flavor composition. We compile the results in Table 3. The table contains also a comparison with the experimental values, to the extent the latter are known. It is clear why one should exempt the quasi-pions from a detailed comparison with data, but for the rest it was not really anticipated that they would agree on the level of about one part in hundred. Some of these mesons have not yet been found. The present model predicts then for example

$$M(B_c^\pm) = 6494\,\mathrm{MeV}\ , \qquad M(B_c^{*\pm}) = 6501\,\mathrm{MeV}\ . \tag{58}$$

Such a table was not given before. In all due respect for the work with lattice gauge theory, the

agreement of the above equations with data is not so much worse, particularly in view of their simplicity and the absence of any free parameters.

6 Summary and Conclusions

Applying DLCQ to QCD in the light-cone gauge $A^+ = 0$ and disregarding *zero modes*, one derives first an effective interaction acting in the $q\bar{q}$ space. We emphasize that no assumption on a possible smallness of the coupling constant has to be made, nor that it is necessary to truncate the Fock space in the manner of Tamm [22] and of Dancoff [23]. Gauge invariance, therefore, is *not* violated. Once this is achieved, one can drop discretization and convert the matrix into an integral equation, Eq.(11). Unfortunately we cannot be more explicit in the present context. We therefore consider this equation as part of the model and the point of departure in the present work. – We discover next a simple substitution of integration variables, Eq.(18), which allows to transmute the equation identically into Eq.(25), in which appear only the conventional instant form variables like the usual momenta and energies. One wishes to Fourier transform the still front form equations to usual space coordinates but is hampered by the 'square roots'. A poor man's solution is their consistent expansion up to second order. The prize to pay is Eq.(49), *i.e.* a smallness requirement *a posteriori* which has to be met in the solution. One ends up with a differential equation in configuration space, Eq.(50), which looks like a non-relativistic Schrödinger equation without being one. It contains the full retardation of a relativistically correct front form equation, and therefore is referred to as the Retarded Schrödinger Equation. Its solutions can be Fourier transformed back to momentum space and subjected to the transmutation equation (18), to get the wavefunction expressed in light-cone momenta particularly longitudinal momentum fractions, which are needed so urgently for the theoretical predictions of experimental high-energy cross sections, and of structure functions and the like.

One should emphasize that the Retarded Schrödinger Equation has no free parameter: The coupling constant and the quark masses have to be determined from the experiment. Like in the familiar quantum electro-dynamics one has no way to calculate them. Fitting the strong coupling constant at the Z-mass and the 5 quark flavor masses to 5 selected vector mesons fixes all our freedom. The rest is structure: The 20 remaining pseudoscalar or vector masses are then predicted and presented in Table 3. In comparison with the experimental data they are not much worse than those from conventional phenomenological models [15, 16], or from heavy quark symmetry

[17], or even from lattice gauge calculations [18, 19, 20]. The pions, of course, remain mysterious particles like in every other model not specially designed for them. One wishes now to solve the full Retarded Schrödinger Equation (50) rather than only its caricature in Eq.(51).

Conclusion: If such a poor model can do so well one must be on the right track. It seems that the front form and DLCQ have made a big step forward. Unfortunately, it took ten years but one has been faced with all the problems of formulating a Hamiltonian approach to gauge field theory, in addition to the problems inherent to the front form.

7 Acknowledgement

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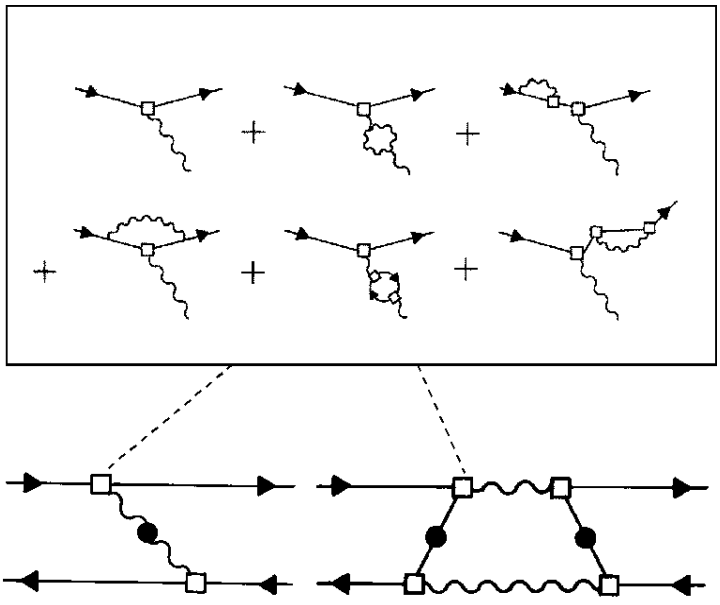

Figure 1: *The effective interaction in the $q\bar{q}$ sector.* — By absorbing (or emitting) an 'effective gluon', a single quark state with four-momentum k_1 and spin-projection s_1 is scattered into the quark state (k_1', s_1'). Correspondingly, the antiquark is scattered from (k_2, s_2) to (k_2', s_2').

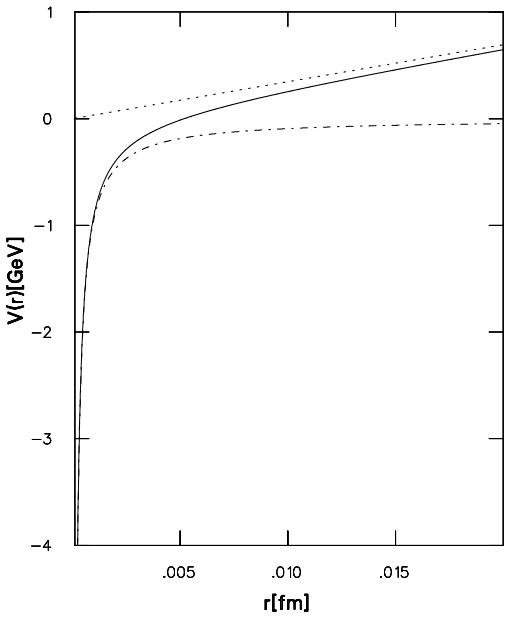

Figure 2: The quark-antiquark potential $V(r)$. *Inserted is also the Coulomb (dashed-pointed) and the confining potential (pointed).*

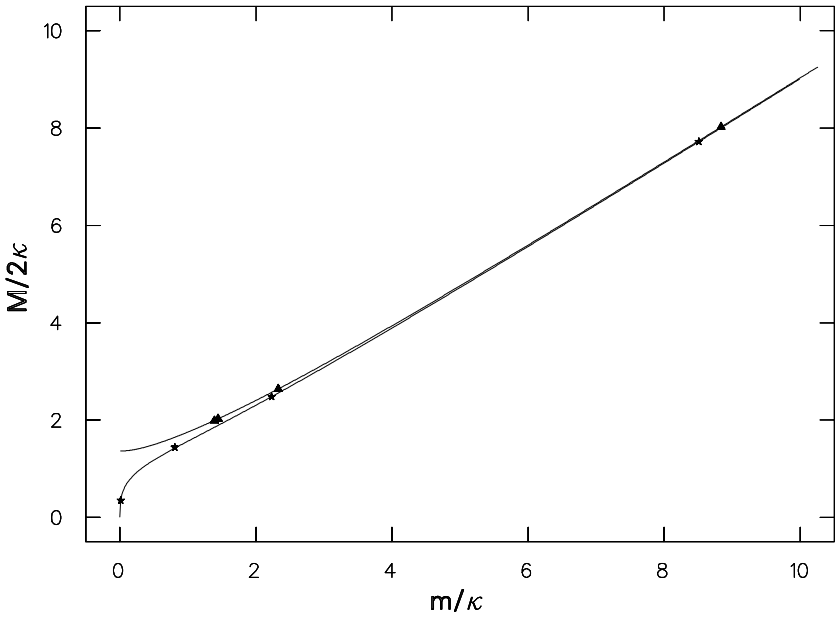

Figure 3: *Bound states of $q\bar{q}$-pairs versus quark masses. — All masses are given in units of the QCD-scale κ. The upper curve refers to the triplet ($S_e = 2$), the lower to the singlet ($S_e = 0$). The masses of some vectormesons ($\rho^0, \omega, \phi, J/\psi$) are marked by ($\triangle$), those of the some pseudoscalars ($\pi^0, \eta, \eta', \eta_c$) by ($\star$).*

Table 3: *The masses of $q\bar{q}$–hadrons are compared with experimental values.* – Flavor quark masses used are inserted in column 2. They in turn come from a fit to the vector mesons, as discussed in the text. – Within each box, the first line refers to the hadronic symbol of the meson; the second line gives the calculated (measured) vector mass in MeV; the third line accounts for the calculated (measured) pseudoscalar mass in MeV, and finally the fourth line specifies the hadronic symbol of the pseudoscalar meson. – Note: The smallness parameter $(\lambda/m)^2$ satisfies always the condition set in the text.

	$\mathbf{m_q}$	$\overline{\mathbf{u}}$	$\overline{\mathbf{d}}$	$\overline{\mathbf{s}}$	$\overline{\mathbf{c}}$	$\overline{\mathbf{b}}$
u	222.8	ρ^0 768(768) 714(135) π^0	ρ^+ 773(768)	K^{*+} 910(892)	$\overline{D}^{*0}$ 2110(2007)	B^{*+} 5712(5325)
d	236.2	658(140) π^-	ω 782(782) 668(549) η	K^{*0} 914(896)	D^{*-} 2109(2010)	B^{*0} 5709(5325)
s	427.2	825(494) K^-	831(498) $\overline{K}^0$	ϕ 1019(1019) 953(958) η'	D_s^{*-} 2156(2110)	B_s^{*0} 5735(—)
c	1701.3	2079(1865) D^0	2078(1869) D^+	2131(1969) D_s^+	J/ψ 3097(3097) 3082(2980) η_c	B_c^{*+} 6502(—)
b	5328.2	5701(5278) B^-	5698(5279) $\overline{B}^0$	5726(5375) $\overline{B}_s^0$	6495(—) B_c^-	Υ 9460(9460) 9455(—) η_b